*Journal of Statistical Physics, Vol. 32, No. 3, 1983*

# Example of Diffusion in a Disordered Lorentz Gas

**Bernard Gaveau[1] and Alain Meritet[2]**

*Received November 15, 1982; revised April 5, 1983*

We prove a diffusion law for a disordered Lorentz gas obtained by modification of a model of Gates, Gerst, Kac in Ref. 1, even though the motion is not a Markovian one in the technical sense of the word.

**KEY WORDS:** Diffusion law; disordered Lorentz gas; non-Markovian process.

## 1. INTRODUCTION

We shall treat here the plane motion of a particle in a disordered Lorentz gas which is obtained by an appropriate modification of a model due to Gates, Gerst, and Kac.[1] In general, a Lorentz gas is a fixed *configuration* of the diffusion centers (*by opposition* to a Rayleigh gas) between which a particle diffuses. Generally the motion of this particle is not Markovian, but we try nevertheless to demonstrate that a diffusion law is true in the sense that

$$\langle q_t^2 \rangle - \langle q_t \rangle^2 \sim Ct \qquad \text{if} \quad t \to +\infty \tag{1}$$

where $q_t$ is the displacement of the particle for $t \to +\infty$. We shall begin by recalling the results of Ref. 1. Then we shall define a family of random Lorentz gas models with a variable concentration $\beta$. We shall see that, as the concentration $\beta$ tends to 0, the constant $C$ diverges like $k/\beta$ (with $k$ a constant).

[1] Mathématiques, Université de Paris VI, 4, Place Jussieu, 75230 Paris, Cedex 05.
[2] INRIA, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay Cedex.

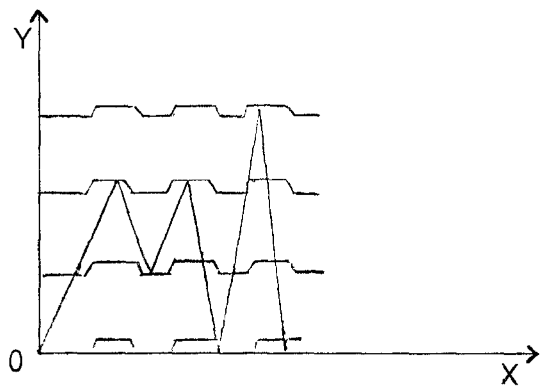


0022-4715/83/0900-0539$03.00/0 

## 2. THE MODEL OF GATES, GERST, KAC

We consider the plane on which we have placed an infinite family of contiguous mirrors: these mirrors are all parallel to the $0x$ axis, with integer ordinate length $1/2$. We denote by $x(m)$ the abscissa of the left end point of the mirror $m$. We make the following rule:

1. If $x(m)$ is an integer, the mirror $m$ allows the particles coming from below to pass without deflection and reflects the particles coming from above.
2. If $x(m)$ is a half integer, the mirror $m$ allows the particles coming from above to pass without deflection and reflects the particles coming from below.




The particle diffuses with a horizontal speed $V > 0$ along the $0x$ axis and a vertical speed at time $t$: $p_t(V) = \pm 1$. ($p_0 = +1$.)

Evidently $V$ does not change. The position is $Q_t(V) = (Vt, q_t(V))$ and we have

$$q_t(V) = \sum_{s=0}^{t-1} p_s(V) \tag{2}$$

We verify easily that $p_t(V) = (-1)^{[2tV]}$, where $[a]$ is the entire part of $a$. We denote by $\langle\ \rangle$ the average on $\mathbb{R}^+$

$$\langle f(V)\rangle = \lim_{V\to+\infty} \frac{1}{V}\int_0^V f(\xi)\,d\xi \tag{3}$$

A fundamental identity is that

$$\langle p_t(V)p_s(V)\rangle = \frac{(t,s)^2}{ts}\chi(t,s)$$

where $(t,s)$ is the greatest common denominator of $t, s$ and $\chi(t,s)$ is defined by

$$\chi(t,s) = \begin{cases} 1 & \text{if 2 divides } t \text{ and } s \text{ the same number of times} \\ 0 & \text{otherwise} \end{cases}$$

This follows from the Fourier expansion of $(-1)^{[2sV]}$ in the series

$$\sum_{n=(2k+1)s} \frac{4s}{\pi n} \sin 2\pi n V$$

For this model, it is demonstrated in Ref. 1 that

$$\langle q_t^2 \rangle - \langle q_t \rangle^2 \sim Ct \qquad \text{if} \quad t \to +\infty$$

where $C$ is a constant explicitly determined.

## 3. INTRODUCTION OF THE DISORDER IN THE PREVIOUS MODEL

For each mirror $m$, we introduce an independent random variable $X_m(\omega)$ taking values 0 or 1 with probability

$$\text{Prob}(X_m = 0) = \alpha$$

$$\text{Prob}(X_m = 1) = \beta \qquad (\alpha + \beta = 1)$$

When $\omega$ is fixed, we have a family $(X_m(\omega))_m$. The mirror $m$ operates if and only if $X_m(\omega) = 1$. This is equivalent to removing in the original model each mirror with probability $\alpha$. When $\alpha = 0$, we are in the Gates–Gerst–Kac case. When $\alpha = 1$, no mirror operates, the particle is not deflected $p_t(V) = p_0(V) = +1$. In the case $0 < \alpha < 1$, we have a disordered Lorentz gas with concentration $\beta = 1 - \alpha$. The notations concerning $p_t(V)$, $Q_t(V)$, $q_t(V)$ are the same as previously. $\langle\ \rangle_\alpha$ denotes now the average on $V$ combined with the mathematical expectation on the configuration $\omega$ of the gas (with the concentration $\beta = 1 - \alpha$)

$$\langle f(\ ) \rangle_\alpha = E_\alpha\left( \lim_{V \to +\infty} \frac{1}{V} \int_0^V f(\xi)\, d\xi \right)$$

We shall show here the following result: put

$$\varphi(t,\alpha) = \langle q_t^2 \rangle_\alpha - \langle q_t \rangle_\alpha^2 \tag{4}$$

Then we have if $t \to +\infty$

$$C_1(\alpha) t \leqslant \varphi(t,\alpha) \leqslant C_2(\alpha) t$$

$$C_1(\alpha) \sim \frac{2\alpha}{1-\alpha} \qquad \text{if} \quad \alpha \to 1^- \tag{5}$$

$$C_2(\alpha) \sim \frac{2\alpha}{1-\alpha} \qquad \text{if} \quad \alpha \to 1^-$$

$k$ being a bounded constant when $\alpha \to 1^-$ which we shall estimate later.

## 4. PRELIMINARY CALCULATION OF $q_t(V)$

We begin by the estimation of $p_s(V)$.

First case: $[2sV]$ is odd. In this case we arrive at time $s$ on a mirror oriented downward. If at time $s-1$, the vertical speed was positive, it becomes negative when the mirror operates and it remains positive if it does not operate. If at time $s-1$ the vertical speed was negative, it does not change regardless of the mirror position.

Second case: $[2sV]$ is even. We arrive then at time $s$ on a mirror oriented upward. If the vertical speed at time $s-1$ was negative, it becomes positive if the mirror operates, negative otherwise. If the vertical speed was positive at time $s-1$, it remains the same in any case.

We denote by $m(Q_s(V))$ the mirror situated at position $Q_s(V)$. We have then the following rule from the previous discussion:

$$p_s(V) = (-1)^{[2sV]X_{m(Q_s(V))}(\omega)}\left[p_{s-1}(V)\right]^{1+X_{m(Q_s(V))}(\omega)} \tag{6}$$

We then remark that the particle never touches the same mirror twice if $V > 1/4$; we shall suppose that $V > 1/4$. By definition of $\langle\ \rangle_\alpha$, as we are concerned only with large $V$, we can always suppose that. Let $\mathscr{B}_{s-1}$ be the $\sigma$ algebra generated by $X_{m(Q_l(V))}(\omega)$ for $l \leqslant s-1$; we have

$$E(p_s(V) \mid \mathscr{B}_{s-1}) = \alpha p_{s-1}(V) + \beta(-1)^{[2sV]}$$

and iterating, we obtain

$$E(p_k(V) \mid \mathscr{B}_l) = p_l(V)\alpha^{k-l} + \beta \sum_{u=l+1}^{k} \alpha^{k-u}(-1)^{[2uV]} \tag{7}$$

$$E(p_l(V)) = \alpha^l + \beta \sum_{v=1}^{l} \alpha^{l-v}(-1)^{[2vV]} \tag{8}$$

It follows that (for $k > l$)

$$\begin{aligned}
E(p_k(V)p_l(V)) &= E(E(p_k(V) \mid \mathscr{B}_l)p_l) \\
&= E\left(p_l(V)^2\alpha^{k-l} + \beta p_l(V) \sum_{u=l+1}^{k} \alpha^{k-u}(-1)^{[2uV]}\right) \\
&= \alpha^{k-l} + \beta\left\{\alpha^l + \beta \sum_{v=1}^{l} \alpha^{l-v}(-1)^{[2vV]}\right\} \\
&\quad \times \left[\sum_{u=l+1}^{k} \alpha^{k-u}(-1)^{[2uV]}\right]
\end{aligned}$$

Thus

$$\langle p_k(V)p_l(V)\rangle_\alpha = \alpha^{k-l} + \beta^2 \sum_{v=1}^{l} \sum_{u=l+1}^{k} \alpha^{l-v}\alpha^{k-u}\psi(v,u)$$

where

$$\psi(v,u) = \langle p_v(V)p_u(V)\rangle_0$$
$$= \left\langle (-1)^{[2vV]}(-1)^{[2uV]}\right\rangle = \frac{(u,v)^2}{uv}\chi(u,v)$$

and

$$\chi(u,v) = \begin{cases} 1 & \text{if 2 divides } u \text{ and } v \text{ the same number of times} \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\langle q_s^2(V)\rangle_\alpha$$
$$= s + 2 \sum_{0\leqslant l<k\leqslant s-1} \langle p_k(V)p_l(V)\rangle_\alpha$$
$$= s + 2 \sum_{0\leqslant l<k\leqslant s-1} \alpha^{k-l} + 2\beta^2 \sum_{1\leqslant l<k\leqslant s-1} \sum_{v=1}^{l} \sum_{u=l+1}^{k} \alpha^{l-v}\alpha^{k-u}\psi(v,u)$$
$$= s + 2 \sum_{0\leqslant l<k\leqslant s-1} \alpha^{k-l} + 2\beta^2 \sum_{1\leqslant v<u\leqslant s-1} \psi(v,u) \sum_{l=v}^{u-1} \alpha^{l-v} \sum_{k=u}^{s-1} \alpha^{k-u}$$

Therefore,

$$\langle q_s^2\rangle_\alpha \geqslant sC_1(\alpha,s) \tag{9}$$

where

$$C_1(\alpha,s) = 1 + \frac{2\alpha}{1-\alpha} + O(s)$$

Similarly,

$$\langle q_s^2\rangle_\alpha \leqslant s\left(1 + \frac{2\alpha}{1-\alpha}\right) + 2\beta^2\left[\sum_{1\leqslant v<u\leqslant s-1} \psi(v,u)\right]\left(\sum_{j=0}^{\infty}\alpha^j\right)\left(\sum_{i=0}^{\infty}\alpha^i\right)$$
$$= s\left(1 + \frac{2\alpha}{1-\alpha}\right) + 2(1-\alpha)^2\frac{1}{(1-\alpha)^2} \sum_{1\leqslant v<u\leqslant s-1} \psi(v,u)$$
$$= s\left(1 + \frac{2\alpha}{1-\alpha}\right) + C(s)$$

where

$$C(s) = 2 \sum_{1\leqslant v<u\leqslant s-1} \psi(v,u) \leqslant \text{const } s$$

by the result of Gates, Gerst, and Kac, or directly by

$$\sum_{1\leqslant v<u\leqslant s-1} \psi(v,u) \leqslant \sum_{1\leqslant v<u\leqslant s-1} \frac{(u,v)^2}{uv} \leqslant \sum_{1\leqslant v'<u'\leqslant s-1} \frac{s}{u'^2v'}$$
$$= \frac{\pi^2}{6}s + O(s)$$

The second inequality follows from the "change of variables"

$$u = du', \qquad v = dv', \qquad (u', v') = 1$$

We have thus proved the result announced in Section 3.

## 5. REMARKS

In fact, if the limit

$$\lim_{V \to \infty} \frac{1}{V} \int_0^V f(\xi)\, d\xi$$

exists for almost every $\omega$, this limit is a constant because it is a tail function for the family $X_m(\omega)$ and so we can apply 0–1 law. In this case if the limits

$$\lim_{V \to +\infty} \frac{1}{V} \int_0^V q_t(\xi)\, d\xi \quad \text{and} \quad \lim_{V \to +\infty} \frac{1}{V} \int_0^V q_t^2(\xi)\, d\xi$$

exist, then they are constants and the results are valid for almost every gas configuration.

## ACKNOWLEDGMENT

The authors are grateful to the referee for a great simplification in the proof of the diffusion law.

## REFERENCES

1. Gates, Gerst, and M. Kac, Non-Markovian Diffusion on Idealized Lorentz Gas, *Ark. Rat. Mech. Anal.* **1973**:106–135.
2 B. Gaveau and A. Méritet, Un Exemple de Diffusion dans un Gaz de Lorentz Désordonné, *C. R. Acad. Sci. Paris* **294**:459 (1982).
3. J. M. Ziman, *Models of Disorder* (Cambridge University Press, Cambridge, 1979).